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OSCILLATORY CONFLICT-CONTROL PROCESSES†

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Some quasi-linear dynamical processes functioning under conditions of conflict [1, 3–8] are considered, on the assumption that Pontryagin's condition [1] holds only in certain intervals of the real half-line (this may occur, in particular, when a homogeneous system is performing periodic oscillations [2]). The method of resolvent functions [3, 4] is used to establish sufficient conditions for the group pursuit problem [3, 4] to be solvable. A typical special case is examined and the group pursuit problem is solved for a second-order system [6]. The results have a bearing on the research reported in [3–5].

1. STATEMENT OF THE PROBLEM

SUPPOSE the state of a process $z=(z_1, \ldots, z_\nu)$, $z_i \in R^{n_i}$, in the space R^n is described by the differential equations

$$\dot{z}_i = A_i z_i + \varphi_i(u_i, \upsilon), \quad u_i \in U_i, \quad \upsilon \in V \tag{1.1}$$

where A_i are square matrices of order n_i, $\varphi_i(u_i, \upsilon)$ are jointly continuous vector-valued functions, and U_i and V are non-empty compact sets ($i=1, 2, \ldots, \nu$ throughout, unless stated otherwise).

The terminal set M is the union of sets $M_1^*, \ldots, M_\nu^*$, each of which can be expressed as $M_i^* = M_i^0 + M_i$, where M_i^0 are linear subspaces of R^{n_i}, and M_i are convex closed sets in the L_i-orthogonal complements of M_i^0 in R^{n_i}.

A trajectory of the conflict-controlled process (1.1) in state $z^0=(z_1^0, \ldots, z_\nu^0)$ may be brought to the terminal set M at an instant of time $T(z^0)$ if measurable functions $u_i(t)=u_i(z_i^0, \upsilon_i(\cdot))$ exist, where $\upsilon_t(\cdot)=\{\upsilon(s): s\in[0, t)\}$, $t\in[0, T(z^0)]$, with values in U_i, such that for at least one i: $z_i(T(z^0)) \in M_i^*$ and any measurable function $\upsilon(t)$, $t\in[0, T(z^0)]$, it is true that $\upsilon(t)\in V$.

Our goal is to establish sufficient conditions, in terms of the parameters of process (1.1), to guarantee that the problem of bringing a trajectory to the terminal set in finite time is solvable.

2. AUXILIARY RESULTS

The proofs of the following results, which we shall need later, may be found in [8–12].

Let $K(R^n)$ be the space of all non-empty compact sets in R^n. We will define a Hausdorff metric in this space [9].

If X, $Y \subset K(R^n)$ and S is the unit sphere about zero in R^n, then $\operatorname{dist}(X, Y) = \min\{\lambda \geq 0: X \subset Y+\lambda S,\ Y \subset X+\lambda S\}$.

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A multiple-valued map $A(x)$, $A\colon X \to K(R^n)$, $X \subset \operatorname{dom} A = \{x,\ A(x) \neq 0\}$ is upper semi-continuous at a point $x_0 \in X$, if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that if $\|x - x_0\| \leq \delta$, then $A(x) \subset A(x_0) + \varepsilon S$. If a map $A(x)$ is upper semi-continuous at each point of a set X, it is said to be upper semi-continuous on X. Given a set X, $X \subset K(R^n)$, we define the cone $\operatorname{con} X = \{z\colon z = \lambda x,\ x \in X,\ \lambda > 0\}$, and let $\overline{\operatorname{con}} X$ denote the closure of $\operatorname{con} X$.

Lemma 1 [8]. Let X, Y, $M \subset K(R^n)$; assume that $A(x, y)$, $A\colon X * Y \to K(R^n)$, is an upper semi-continuous (multiple-valued) map and $f(x)$, $f\colon X \to R^n$, a continuous function such that $f(x) \cap M = \varnothing$ for any $x \in X$, $y \in Y$. Then the function $\alpha(x, y)$, $\alpha\colon X * Y \to R^1$, defined by $\alpha(x, y) = \max\{\alpha \geq 0\colon \alpha(M - f(x)) \cap A(x, y) \neq 0\}$ is upper semi-continuous.

Lemma 2 [10]. Let $X \subset K(R^n)$; assume that $T(x)$, $T\colon X \to K(R^n)$, $A(x)$ $A\colon X \to K(R^n)$ are upper semi-continuous (multiple-valued) maps and $f(x, y)$, $x \in X$, $y \in A(x)$, $f(x, y) \in R^n$ is a continuous function. Then the multiple-valued map $C(x) = \{y \in A(x)\colon f(x, y) \in T(x)\}$ is upper semi-continuous.

We shall say that a multiple-valued map $A(x)$, $A\colon X \to K(R^n)$, is Lebesgue (Borel) measurable if X is a Lebesgue- (Borel-)measurable set and, for any $Y \subset K(R^n)$, the set $\{x \in X\colon A(x) \subset Y\}$ is Lebesgue (Borel) measurable. To simplify the terminology, we shall call Lebesgue-measurable maps simply measurable, and refer to Borel-measurable maps as Borel maps.

Lemma 3 [11]. Let $X \subset K(R^n)$; assume that $T(x)$, $T\colon X \to K(R^n)$, $A(x)$, $A\colon X \to K(R^n)$, are measurable (Borel) multiple-valued maps and that the function $f(x, y)$, $x \in X$, $y \in A(x)$, $f(x, y) \in R^n$, is measurable (Borel) as a function of x and continuous as a function of y. Then the multiple-valued map $C(x) = \{y \in A(x)\colon f(x, y) \in T(x)\}$ is measurable (Borel).

Let $X \subset K(R^n)$; let X_1 be the set of vectors $x \in X$ whose least component is their first one, X_2 the set of $x \in X_1$ whose least component is the second one, and so on, up to X_n. The set X_n clearly consists of a single point x^*. Then x^* is called the lexicographic minimum of X; let $x^* = \operatorname{lexmin} X$.

A selector of a multiple-valued map $A(x)$, $A\colon X \to K(R^n)$, is a single-valued function $a(x)$ such that $a(x) \in A(x)$ for all $x \in X$.

Lemma 4 [12]. Let $X \subset K(R^n)$, and let $A(x)$, $A\colon X \to K(R^n)$ be a measurable (Borel) map. Then the selector $a(x) = \operatorname{lexmin} A(x)$, $x \in X$ is measurable (Borel).

Lemma 5 [9]. Let X, Y, $Z \subset K(R^n)$; let $\varphi(y)$, $\varphi\colon Y \to Z$ be a Borel function and $y(x)$, $y\colon X \to Y$, a measurable function. Then the function $\psi(t) = \varphi(y(x))$, $\psi\colon X \to Z$ is measurable.

3. SCHEME OF THE METHOD

Let π_i denote the orthogonal projection operator from R^n on to the subspace L_i. Using the functions $W_i(t, u_i, \upsilon) = \pi_i \Phi_i(t) \varphi_i(u_i, \upsilon)$, $t \geq 0$, $u_i \in U_i$, $\upsilon \in V$ (where $\Phi_i(t) = \exp(tA_i)$), we define multiple-valued maps

$$W_i(t, \upsilon) = \bigcup_{u_i \in U_i} W_i(t, u_i, \upsilon), \quad W_i(t) = \bigcap_{\upsilon \in V} W_i(t, \upsilon)$$

Pontryagin's condition means that $W_i(t) \neq 0$ for all $t \geq 0$. We shall adopt certain rather weaker assumptions [13].

Condition 1.

$$\operatorname{dom} W_i(t) = \left\{ \bigcup_{k=0}^{\infty} [t_{2k}^i, t_{2k+1}^i] \right\}, \quad t_0^i = 0, t_j^i < t_{j+1}^i$$

for all $j = 0, 1, 2, \ldots$.

Put

$$\Delta_+^i = \bigcup_{k=0}^{\infty} [t_{2k}^i, t_{2k+1}^i], \quad \Delta_-^i = \bigcup_{k=0}^{\infty} (t_{2k+1}^i, t_{2k+2}^i)$$

Condition 2. Borel multiple-valued maps $Q_i(t)$, $Q_i : \Delta_-^i \to K(L_i)$ exist, such that
1. we have

$$\bigcap_{\upsilon \in V} \{W_i(t, \upsilon) + Q_i(t)\} \neq \varnothing$$

for all $t \in \Delta_-^i$ and
2. we have

$$\int_{t_{2k+1}^i}^{t_{2k+2}^i} Q_i(\tau)d\tau \subset \int_{t_{2k}^i}^{t_{2k+1}^i} W_i(\tau)d\tau$$

for all $k = 0, 1, 2, \ldots$.
Define times

$$\bar{t}_{2k+1}^i = \max\left[t \leqslant t_{2k+1}^i : \int_{t_{2k+1}^i}^{t_{2k+2}^i} Q_i(\tau)d\tau \subset \int_t^{t_{2k+1}^i} W_i(\tau)d\tau\right] \tag{3.1}$$

$k = 0, 1, 2, \ldots$.
Fix $t \in [0, +\infty)$. For every i there exists an integer $p_i \geqslant 0$ such that $t \in [t_{2p_i}^i, t_{2p_i+1}^i]$ or $t \in (t_{2p_i+1}^i, t_{2p_i+2}^i)$.
For i such that $t \in [t_{2p_i}^i, t_{2p_i+1}^i]$, we define sets $\Delta_-^i(t)$, $\Delta_0^i(t)$, $\tilde{\Delta}_+^i(t)$ by

$$\Delta_-^i(t) = \bigcup_{k=0}^{p_i-1} (t - t_{2k+2}^i, t - t_{2k+1}^i); \quad \Delta_0^i(t) = \bigcup_{k=0}^{p_i-1} [t - t_{2k+1}^i, t - \bar{t}_{2k+1}^i]$$

$$\tilde{\Delta}_+^i(t) = \bigcup_{k=0}^{p_i-1} (t - \bar{t}_{2k+1}^i, t - t_{2k}^i) \cup [0, t - t_{2p_i+1}^i]$$

For i such that $t \in (t_{2p_i+1}^i, t_{2p_i+2}^i)$, we define sets $\Delta_0^i(t)$, $\Delta_-^i(t)$, $\tilde{\Delta}_+^i(t)$ by

$$\Delta_-^i(t) = \bigcup_{k=0}^{p_i-1} (t - t_{2k+2}^i, t - t_{2k+1}^i) \cup [0, t - t_{2p_i+1}^i]$$

$$\Delta_0^i(t) = \bigcup_{k=0}^{p_i} [t - t_{2k+1}^i, t - \bar{t}_{2k+1}^i]; \quad \tilde{\Delta}_+^i(t) = \bigcup_{k=0}^{p_i-1} (t - \bar{t}_{2k+1}^i, t - t_{2k}^i)$$

For fixed t, $t > 0$, we let

$$\Gamma_i(t) = \left\{\gamma_i(\cdot) : \begin{array}{l} \gamma_i(t-\tau) \in W_i(t-\tau), \tau \in \tilde{\Delta}_+^i(t) \\ \gamma_i(t-\tau) = 0, \tau \in [0,t] \setminus \tilde{\Delta}_+^i(t) \end{array}\right\}$$

denote the set of Borel selectors of the map $W(t-\tau)$, $t \geqslant \tau \geqslant 0$. Set $\gamma(\cdot) = \gamma_1(\cdot), \ldots, \gamma_\nu(\cdot))$, $\Gamma(\cdot) = (\Gamma_1(\cdot), \ldots, \Gamma_\nu(\cdot))$.
Fixing some Borel selector $\gamma(\cdot) \in \Gamma(t)$, we put

$$\xi_i(t, z_i, \gamma_i(\cdot)) = \pi_i \Phi_i(t) z_i + \int_0^t \gamma_i(t-\tau)d\tau \tag{3.2}$$

We now define the resolvent functions

$$\mu_i(t,\tau,z_i,\upsilon,\gamma_i(\cdot)) = \begin{cases} \sup[\mu \geq 0: W_i(t-\tau,\upsilon) - \gamma_i(t-\tau) \cap \mu(M_i - \xi_i(t,z_i,\gamma_i(\cdot))) \neq \phi] \\ \tau \in \tilde{\Delta}^i_+(t) \\ 0, \tau \in [0,t] \setminus \tilde{\Delta}^i_+(t) \end{cases} \tag{3.3}$$

Set

$$\mu(t,\tau,z,\upsilon,\gamma(\cdot),\alpha) = \sum_{i=1}^{\nu} \alpha_i \mu_i(t,\tau,z_i,\upsilon,\gamma_i(\cdot))$$

$$\alpha \in U = \left\{\alpha: \alpha = (\alpha_1,\dots,\alpha_\nu), \alpha_i \geq 0, \sum_{i=1}^{\nu} \alpha_i = 1\right\}$$

and define a time

$$T(z,\gamma(\cdot)) = \min\left\{t \geq 0: 1 - \inf_{\upsilon(\cdot)\in\Omega_V} \max_{\alpha\in U} \int_0^t \mu(t,\tau,z,\upsilon(\tau),\gamma(\cdot),\alpha)d\tau \leq 0\right\} \tag{3.4}$$

$\Omega_V = \{\upsilon(\cdot): \upsilon(\tau) \in V, \tau \geq 0, \upsilon(\tau)$ is a measurable function$\}$.

If $\xi_i(t, z_i, \gamma_i(\cdot)) \notin M_i$, the resolvent function $\mu(t, \tau, z, \upsilon, \gamma(\cdot), \alpha)$ is finite for any values of the arguments, and by Lemma 1 it is Borel with respect to υ, τ, t. Consequently, $\mu(t, \tau, z, \upsilon, \gamma(\cdot), \alpha)$ is an integrable function in any finite interval.

If an i exists such that at time t^* we have $\xi_i(t^*, z_i, \gamma_i(\cdot)) \in M_i$ and $\alpha_i \neq 0$, then $\mu(t^*, \tau, z, \upsilon, \gamma(\cdot), \alpha) = +\infty$ for any τ, υ. Using the fact that the integral of a function that equals $+\infty$ in a finite interval is also equal to $+\infty$, we deduce that inequality (3.4) is automatically true, so that $t^* = T(z, \gamma(\cdot))$.

4. MAIN THEOREM

Theorem 1. Suppose that the conflict-controlled process (1.1) is in its initial state z^0 and that conditions 1 and 2 are satisfied; suppose, moreover, that Borel selectors $\gamma_i^0(t-\tau)$, $\gamma_i^0(t-\tau) \in \Gamma_i(t)$, $t \geq \tau \geq 0$ exist, such that $T(z^0, \gamma^0(\cdot)) < +\infty$. Then the trajectory of the process may be brought to the terminal set M at time $T(z^0, \gamma^0(\cdot))$.

Proof. Put $T(z^0, \gamma^0(\cdot)) = T$. Let $\upsilon(\tau) \in \Omega_V$.

Let us assume that $\xi_i(T, z_i^0, \gamma_i^0(\cdot)) \notin M_i$ for all $i = 1, 2, \dots, \nu$. Define the test function as follows:

$$\sigma(T,t,z^0,\upsilon(\cdot),\gamma^0(\cdot)) = 1 - \max_{\alpha\in U} \int_0^t \mu(T,\tau,z^0,\upsilon(\tau),\gamma^0(\cdot))d\tau$$

Since $\sigma(T, 0, z^0, \upsilon(\cdot), \gamma^0(\cdot)) = 1$ and $\sigma(T, t, z^0, \upsilon(\cdot), \gamma^0(\cdot))$ is a continuous decreasing function of t, it follows from (3.4) that a time $t_*: 0 < t_* \leq T$ exists such that $\sigma(T, t_*, z^0, \upsilon(\cdot), \gamma^0(\cdot)) = 0$.

We choose controls $u_i(\tau)$, $u_i(\tau) \in U_i$ for $\tau \in [0, t_*]$ as follows.

1. Let $\tau \in \tilde{\Delta}^i_+(T) \cap [0, t_*]$.

Consider the multiple-valued map defined by

$$U_i^1(\tau,\upsilon) = \{u_i \in U_i: W_i(T-\tau,u_i,\upsilon) - \\ - \gamma_i^0(T-\tau) \in \mu_i(T,\tau,z_i^0,\upsilon,\gamma_i^0(\cdot))(M_i - \xi_i(T,z_i^0,\gamma_i^0(\cdot)))\}$$

Remembering our assumptions about the parameters of the process (1.1), we may conclude that $W_i(T-\tau,\ u_i,\ \upsilon)-\gamma_i^0(T-\tau)$ is a Borel function of τ and a continuous function of u_i, and that the multiple-valued function

$$\mu_i(T,\tau,z_i^0,\upsilon(\tau),\gamma_i^0(\cdot))[M_i-\xi_i(T,z_i^0,\gamma_i^0(\cdot))]$$

is a Borel function of τ, υ, since by Lemma 1 $\mu_i(T,\ \tau,\ z_i^0,\ \upsilon,\ \gamma_i^0(\cdot))$ is an upper semi-continuous function of υ.

By Lemma 3, $U_1^i(\tau,\ \upsilon)$ is a Borel function of υ, τ. Starting from the multiple-valued map $U_1^i(\tau,\ \upsilon)$ we consider the selector $u_1^i(\tau,\ \upsilon)=\operatorname{lexmin}U_1^i(\tau,\ \upsilon)$.

By Lemma 4, $u_1^i(\tau,\ \upsilon)$ is a Borel function of τ, υ.

We now define the control $u_1^i(\tau)$ for $\tau\in\tilde{\Delta}_+^i(T)\cap[0,\ t_*]$ to be $u_i(\tau)=u_1^i(\tau,\ \upsilon(\tau))$. Then, by Lemma 5, $u_i(\tau)$ is measurable.

2. Let $\tau\in\Delta_-^i(T)$. We form the multiple-valued map

$$U_2^i(\tau,\upsilon)=\{u_i\in U_i:W_i(T-\tau,u_i,\upsilon)\in -Q_i(T-\tau)\}$$

By condition 2 and Lemmas 2 and 3, $U_2^i(\tau,\ \upsilon)$ is a non-empty Borel function of τ and an upper semi-continuous function of υ.

Define $u_2^i(\tau,\ \upsilon)=\operatorname{lexmin}U_2^i(\tau,\ \upsilon)$ and define the control $u_i(\tau)$ for $\tau\in\Delta_-^i(T)$ to be $u_2^i(\tau,\ \upsilon(\tau))$.

As in case 1, one shows that $u_i(\tau)$ is a measurable function of τ for $\tau\in\Delta_-^i(T)$.

Put

$$\eta_{2k+1}^i(u_i(\cdot),\ \upsilon(\cdot))=\int_{T-t_{2k+2}^i}^{T-t_{2k+1}^i}W_i(T-\tau,\ u_i(\tau),\ \upsilon(\tau))\,d\tau,\quad k=p_i-1,\ \ldots,0.$$

For i such that $t\in(t_{2p_i+1}^i,\ t_{2p_i+2}^i)$, if $k=p_i$, we obtain

$$\eta_{2p_i+1}^i(u_i(\cdot),\ \upsilon(\cdot))=\int_0^{T-t_{2p_i+1}^i}W_i(T-\tau,\ u_i(\tau),\ \upsilon(\tau))\,d\tau.$$

3. Let $\tau\in\Delta_0^i(T)$. Then $\tau\in[t-t_{2k+1}^i,\ t-\tilde{t}_{2k+2}^i]$, where $k=p_i-1,\ \ldots,0$ for i such that $T\in[t_{2p_i}^i,\ t_{2p_i+1}^i]$, and $k=p_i,\ \ldots,0$ for i such that $T\in(t_{2p_i+1}^i,\ t_{2p_i+2}^i)$.

By the definition of $u_i(\tau)$, for $\tau\in\Delta_-^i(T)$

$$\begin{aligned}-\eta_{2k+1}^i(u_i(\cdot),\ \upsilon(\cdot))&\in\int_{T-t_{2k+2}^i}^{T-t_{2k+1}^i}Q_i(T-\tau)\,d\tau\\-\eta_{2p_i+1}^i(u_i(\cdot),\ \upsilon(\cdot))&\in\int_0^{T-t_{2p_i+1}^i}Q_i(T-\tau)\,d\tau\end{aligned}\tag{4.1}$$

It follows from (3.1) and (4.1) that

$$-\eta_{2k+1}^i(u_i(\cdot),\ \upsilon(\cdot))\in\int_{T-t_{2k+1}^i}^{T-\tilde{t}_{2k+1}^i}W_i(T-\tau)\,d\tau\tag{4.2}$$

By (4.2), a Borel selector $h_{2k+1}^i(T-\tau)$ of the map $W_i(T-\tau)$, $\tau\in(T-t_{2k+1}^i,\ T-\tilde{t}_{2k+1}^i)$ exists, such that

$$\int_{T-t_{2k+1}^i}^{T-\tilde{t}_{2k+1}^i}h_{2k+1}(T-\tau)\,d\tau=-\eta_{2k+1}^i(u_i(\cdot),\ \upsilon(\cdot)).$$

For those i such that $T\in(t^i_{2p_i+1},\ t^i_{2p_i+2})$, we have $k=p_i,\dots,0$.
For those i such that $T\in(t^i_{2p_i},\ t^i_{2p_i+1})$, we have $k=p_i-1,\dots,0$.
Define $h^i(T-\tau)=h^i_{2k+1}(T-\tau)$ for all k.
Thus, the function $h^i(T-\tau)$ has been defined for all $\tau\in\Delta^i_-(t)$. We now form the multiple-valued map

$$U_3^i(\tau,\upsilon)=\{u_i\in U_i: W_i(T-\tau,u_i,\upsilon)=h^i(T-\tau)\}$$

By Lemmas 2 and 3, $U_3^i(\tau,\ \upsilon)$ is a Borel function of τ and an upper semi-continuous function of υ.
Put $u_3^i(\tau,\ \upsilon)=\text{lexmin}\,U_3^i(\tau,\ \upsilon)$, and define the control $u_i(\tau)$ to be $u_3^i(\tau,\ \upsilon(\tau))$.
By Lemmas 4 and 5, we see that $u_i(\tau)$ is a measurable function of τ for $\tau\in\Delta^i_0(T)$.
4. Let $\tau\in\tilde\Delta^i_+(T)\cap[t_*,\ T]$. We form a multiple-valued map

$$U_4^i(\tau,\upsilon)=\{u_i\in U_i: W_i(T-\tau,u_i,\upsilon)=\gamma_i^0(T-\tau)\}$$

Define $u_4^i(\tau,\ \upsilon)=\text{lexmin}\,U_4^i(\tau,\ \upsilon)$, and define the control $u_i(\tau)$ to be $u_4^i(\tau,\ \upsilon(\tau))$.
As in case 3, one shows that $u_i(\tau)$ is a measurable function in the interval $\tau\in\tilde\Delta^i_+(T)\cap[t_*,\ T]$.
By Cauchy's formula

$$\pi_i z_i(T)=\pi_i\Phi_i(T)z_i^0+\int_0^T W_i(T-\tau,\ u_i(\tau),\ \upsilon(\tau))d\tau. \tag{4.3}$$

Taking the definition of the control $u_i(\tau)$ for $\tau\in\Delta^i_-(T)$ and $\tau\in\Delta^i_0(T)$ into account, we obtain

$$\int_{T-t^i_{2k+2}}^{T-t^i_{2k+1}} W_i(T-\tau,\ u_i(\tau),\ \upsilon(\tau))d\tau+\int_{T-\bar t^i_{2k+1}}^{T-\bar t^i_{2k+1}} W_i(T-\tau,\ u_i(\tau),\ \upsilon(\tau))d\tau=0 \tag{4.4}$$

for all $k=p_i-1,\dots,0$.
For i such that $T\in(t^i_{2p_i+1},\ t^i_{2p_i+2})$ when $k=p_i$, we obtain

$$\int_0^{T-t^i_{2p_i+1}} W_i(T-\tau,\ u_i(\tau),\ \upsilon(\tau))d\tau+\int_{T-t^i_{2p_i+1}}^{T-\bar t^i_{2p_i+1}} W_i(T-\tau,\ u_i(\tau),\ \upsilon(\tau))d\tau=0 \tag{4.5}$$

By the method of resolvent functions [3], we see that for $\tau\in\tilde\Delta^i_+(T)$

$$W_i(T-\tau,u_i(\tau),\upsilon(\tau))-\gamma_i^0(t-\tau)\in\mu_i(T,\tau,z_i^0,\upsilon(\tau),\gamma_i^0(\cdot))[M_i-\xi_i(T,z_i^0,\gamma_i^0(\cdot))] \tag{4.6}$$

Taking into account that the functions

$$W_i(T-\tau,u_i(\tau),\upsilon(\tau)),\quad \gamma_i^0(T-\tau),\quad \mu_i(T,\tau,z_i^0,\upsilon(\tau),\quad \gamma_i^0(\cdot))$$

are measurable with respect to τ, we deduce from (4.6) that

$$\int_{\tilde\Delta^i_+(T)} W_i(T-\tau,\ u_i(\tau),\ \upsilon(\tau))d\tau\in[M_i-\xi_i(T,\ z_i^0,\ \gamma_i^0(\cdot))]\int_{\tilde\Delta^i_+(T)}\mu_i(T,\ \tau,\ z_i^0,\ \upsilon(\tau),\ \gamma_i^0(\cdot))d\tau+\int_{\tilde\Delta^i_+(T)}\gamma_i^0(T-\tau)d\tau \tag{4.7}$$

Noting that $\gamma_i^0(T-\tau)=0$ and $\mu_i(T,\ \tau,\ z_i^0,\ \upsilon(\tau),\ \gamma_i^0(\cdot))=0$ for $\tau\in[0,\ T]\setminus\Delta^i_+(T)$, and using (4.4) and (4.5), we can write (4.7) in the form

$$\int_0^T W_i(T-\tau, u_i(\tau), \upsilon(\tau))d\tau \in [M_i - \xi_i(T, z_i^0, \gamma_i^0(\cdot))] \int_0^T \mu_i(T, \tau, z_i^0, \upsilon(\tau), \gamma_i^0(\cdot))d\tau + \int_0^T \gamma_i^0(T-\tau)d\tau \tag{4.8}$$

The test function $\sigma(T, T, z^0, \upsilon(\cdot), \gamma(\cdot))$ vanishes by the definition of the controls $u_i(\tau)$, i.e. an index i_0 exists such that

$$1 - \int_0^T \mu_{i_0}(T, \tau, z_{i_0}^0, \upsilon(\tau), \gamma_{i_0}^0(\cdot))d\tau = 0. \tag{4.9}$$

It follows from (3.2), (4.3), (4.8) and (4.9) that

$$\pi_{i_0} z_{i_0}^0(T) \in M_{i_0}$$

Let us consider the case when $\xi_{i_0}(T, z_{i_0}^0, \gamma_{i_0}^0(\cdot)) \in M_{i_0}$ for some number i_0. We then define the control $u_{i_0}(\tau)$, $u_{i_0}(\tau) \in U_i$, $\tau \in [0, T]$ as follows:

$$u_{i_0}(\tau) = \begin{cases} u_2^{i_0}(\tau, \upsilon(\tau)), & \tau \in \Delta_-^{i_0}(T) \\ u_3^{i_0}(\tau, \upsilon(\tau)), & \tau \in \Delta_0^{i_0}(T) \\ u_4^{i_0}(\tau, \upsilon(\tau)), & \tau \in \bar{\Delta}_+^{i_0}(T) \end{cases}$$

It follows from (3.2) and (4.3) that in this case $\pi_{i_0} z_{i_0}^0(T) \in M_{i_0}$ also. This proves the theorem.

5. MODIFIED METHOD

We shall now examine another approach to the solution of our problem.
We introduce multi-valued maps

$$\begin{gathered} W_i(t, \tau, \upsilon) = \pi_i \Phi_i(t-\tau)\varphi_i(U_i, \upsilon) - \omega_i(t, \tau)M_i \\ W_i(t, \tau) = \bigcap_{\upsilon \in V} W_i(t, \tau, \upsilon),\ \omega_i(t, \tau) \geqslant 0,\ \int_0^t \omega_i(t, \tau)d\tau = 1 \end{gathered} \tag{5.1}$$

Condition 3.

$$\operatorname{dom} Wi(t, \tau) = \bigcup_{k=0}^{\infty} \Delta^i(k, t), \quad \text{for all } t \geqslant 0, \tau \in [0, t]$$

$$\Delta^i(k, t) = \begin{cases} [t_{2k}^i, t], & t \in [t_{2k}^i,\ t_{2k+1}^i) \\ [t_{2k}, t_{2k+1}^i), & t \geqslant t_{2k+1}^i \\ \phi, & t < t_{2k}^i \end{cases}$$

Define sets $\Delta_-^i(k, t)$ and $\Delta_+^i(k, t)$ by the formulae

$$\Delta_-^i(k, t) = \begin{cases} (t - t_{2k+2}^i, t - t_{2k+1}^i), & t \geqslant t_{2k+2}^i \\ [0, t - t_{2k+1}^i), & t \in [t_{2k+1}^i,\ t_{2k+2}^i) \\ \phi, & t < t_{2k+1}^i \end{cases} \tag{5.2}$$

$$\Delta_+^i(k,t)=\begin{cases}[t-t_{2k+1}^i, t-t_{2k}^i], & t\geqslant t_{2k+1}^i\\ [0, t-t_{2k}^i], & t\in[t_{2k}^i, t_{2k+1}^i)\\ \phi, & t<t_{2k}^i\end{cases} \tag{5.3}$$

Put $k_i(t)=\max[k\geqslant 0: \Delta_-^i(k,t)\neq 0]$.

Now set

$$\Delta_-^i(t)=\bigcup_{k=0}^{k_i(t)}\Delta_-^i(k,t);\quad \Delta_+^i(t)=\bigcup_{k=0}^{k_i(t)+1}\Delta_+^i(k,t)$$

Condition 4. Borel multi-valued maps $Q_i(t,\tau)$, $Q_i: [0, +\infty]*\Delta_-(t)\to K(L_i)$ exist, such that

1. $\bigcap_{\upsilon\in V}\{W_i(t,\tau,\upsilon)+Q_i(t,\tau)\}\neq\varnothing$, for all $\tau\in\Delta_-^i(t)$.
2. $\int_{\Delta_-^i(k,t)}Q_i(t,\tau)d\tau\subset\int_{\Delta_+^i(k,t)}W_i(t,\tau)d\tau$, for all $k=0,\dots,k_i(t)$.

Define the times

$$\tilde t_{2k+1}^i=\max\left[\tilde t\leqslant t_{2k+1}^i: \int_{\Delta_-^i(k,t)}Q_i(t,\tau)d\tau\subset\int_{t-t_{2k+1}^i}^{t-\tilde t}W_i(t,\tau)d\tau\right]$$

Now set

$$\Delta_0^i(k,t)=\begin{cases}[t-t_{2k+1}^i, t-\tilde t_{2k+1}^i], t\geqslant t_{2k+1}^i\\ \phi, t<t_{2k+1}^i\end{cases}\quad \Delta_0^i(t)=\bigcup_{k=0}^{k_i(t)}\Delta_0^i(k,t),\quad \tilde\Delta_+^i(t)=\Delta_+^i(t)\setminus\Delta_0^i(t) \tag{5.4}$$

Put

$$\Gamma_i(t)=\left\{\gamma_i(\cdot):\ \begin{matrix}\gamma_i(t,\tau)\in W_i(t,\tau), & \tau\in\tilde\Delta_+^i(t),\\ \gamma_i(t,\tau)=0, & \tau\in[0,t]\setminus\tilde\Delta_+^i(t),\end{matrix}\quad \gamma_i(\cdot)\text{ is Borel}\right\}$$

Set

$$\xi_i(t, z_i, \gamma_i(\cdot))=\pi_i\Phi_i(t)z_i+\int_0^t\gamma_i(t,\tau)d\tau$$

$$\mu_i(t,\tau,z_i,\upsilon,\gamma_i(\cdot))=\left\{\begin{matrix}\sup[\mu\geqslant 0: -\mu\xi_i(t,z_i,\gamma_i(\cdot))\in W_i(t,\tau,\upsilon)-\gamma_i(t,\tau)]\\ \tau\in\tilde\Delta_+^i(t)\\ 0, \tau\in[0,t]\setminus\tilde\Delta_+^i(t)\end{matrix}\right\}$$

$$\mu(t,\tau,z,\upsilon,\gamma(\cdot),\alpha)=\sum_{i=1}^{v}\alpha_i\mu_i(t,\tau,z_i,\upsilon,\gamma_i(\cdot))$$

$$T_{\alpha(\cdot)}(z,\gamma(\cdot))=\min\left\{t\geqslant 0: 1-\inf_{\upsilon(\cdot)\in\Omega_V}\max_{\alpha\in U}\int_0^t\mu(t,\tau,z,\upsilon(\tau),\gamma(\cdot),\alpha)d\tau\leqslant 0\right\}$$

Theorem 2. Suppose that the conflict-controlled process (1.1) is in state z^0 and that non-negative Borel functions $\omega_i(t,\tau)$, $t\geqslant\tau\geqslant 0$, and Borel selectors $\gamma_i^0(t,\tau)\in\Gamma_i(t)$ exist such that

$$T = T_{\omega(\cdot)}(z^0, \gamma^0(\cdot)) < +\infty, \qquad \int_0^T \omega_i(T, \tau)d\tau = 1.$$

Then the trajectory of process (1.1) may be brought to the terminal set M at time T.
The proof is analogous to that of Theorem 1.

6. SPECIAL CASE

Let us consider the special case in which $\varphi_i(u_i, \upsilon) = u_i - \upsilon$, $U_i = \rho S$, $V = \sigma S$, $M_i = \varepsilon S$, $n_i = n$. Put $\xi_i(t, z_i, 0) = \pi\Phi_i(t)z_i$.

Condition 5. A number $p < +\infty$: $p = \min\{\tilde{p} > 0 : \xi_i(t + \tilde{p}, z_i) = \xi_i(t, z_i), \ \forall z_i \in R^n\}$ exists.

Condition 6.

$$\operatorname{dom} W_i(t, \tau) = \bigcup_{k=0}^{\infty} \Delta(k, t) \quad t \geqslant 0, \tau \in [0, t]$$

$$\Delta(k,t) = \begin{cases} [t_{2k}, t], & t \in [t_{2k}, t_{2k+1}) \\ [t_{2k}, t_{2k+1}), & t \geqslant t_{2k+1} \\ \phi, & t < t_{2k} \end{cases}$$

Condition 7. $\theta \in [0, p]$ exists such that $0 \in \operatorname{int} \operatorname{co} \xi_i(\theta, z_i)$.
Using analogues of formulae (5.2)–(5.4), we define sets

$$\Delta_-(k,t), \quad \Delta_+(k,t), \quad \Delta_0(k,t)$$

Let us write $e_i(t, z_i) = (-\xi_i(t, z_i))(\|\xi_i(t, z_i\|)^{-1}$, provided that $\xi_i(t, z_i) \neq 0$

$$\eta_{2k+1}(t) = \int_{\Delta_-(k,t)} \{\sigma(t-\tau) - \rho(t-\tau) - \omega(t, \tau)\} d\tau, \quad k = k(t), \ldots, 0.$$

Set $Q_{2k+1}(t) = \tilde{\eta}_{2k+1}(t)S$. For $\eta_{2k+1} \in Q_{2k+1}(t)$, define functions

$$\beta^i_{2k+1}(\eta_{2k+1}) = (\tilde{\eta}_{2k+1}(t) - \|\eta_{2k+1}\|)(\|\xi_i(t, z_i)\|)^{-1}$$

Provided that $(\eta_{2k+1}, e_i(t, z_i)) \leqslant 0$, we have

$$\beta^i_{2k+1}(\eta_{2k+1}) = ((\eta_{2k+1}, e_i(t, z_i)) + \tilde{\eta}_{2k+1}(t) - \|\eta_{2k+1} - e_i(t, z_i) \times$$
$$\times (\eta_{2k+1}, e_i(t, z_i))\|)(\|\xi_i(t, z_i)\|)^{-1}, \quad \text{if} \quad (\eta_{2k+1}, e_i(t, z_i)) > 0$$
$$\beta_{2k+1}(\eta_{2k+1}) = \sum_{i=1}^{\nu} \alpha_i \beta^i_{2k+1}(\eta_{2k+1})$$

We form a multi-valued map

$$\Theta(z) = \{\theta : 0 \in \operatorname{int} \operatorname{co} \xi_i(\theta, z_i)\}$$

By condition $\Theta(z) \neq \varnothing$. By condition 5, if $\theta_1 \in \Theta(z)$, then for all $k = 0, 1, \ldots,$ we have $\{\theta_1 + kp\} \in \Theta(z)$.
Define resolvent functions by

$$\mu_i(t,\tau,z_i,\upsilon)=\begin{cases}\sup[\mu_i\geq 0:-\mu_i\xi_i(t,z_i)\in W_i(t,\tau,\upsilon)], & \text{if} \quad \tau\in\tilde{\Delta}_+(t)\\ 0, \quad \text{if} \quad \tau\in[0,t]\setminus\tilde{\Delta}_+(t)\end{cases}$$

$$\mu(t,\tau,\upsilon,\alpha)=\sum_{i=1}^{\nu}\alpha_i\mu_i(t,\tau,z_i,\upsilon)$$

For $t\in\Theta(z)$, we write

$$\lambda(t,z)=1-\inf_{\upsilon(\cdot)\in\Omega_V}\min_{\substack{\eta_{2k+1}\in Q_{2k+1}\\ k=k(t),\dots,0}}\max_{(t)\alpha\in U}\left\{\int_0^t\mu(t,\tau,z,\upsilon(\tau),\alpha)\,d\tau+\sum_{k=0}^{p}\beta_{2k+1}(\eta_{2k+1})\right\}$$

Finally, define a time $T^*_{\omega(\cdot)}(z)=\min\{t\geq 0;\ t\in\Theta(z):\ \lambda(t,z)\leq 0\}$.

Theorem 3. Suppose that the conflict-controlled process (1.1) is in state z^0 and

1. conditions 5 and 7 are satisfied,
2. a non-negative Borel function $\omega(t,\tau)$, $t\geq\tau\geq 0$, exists such that conditions 6 and 4 are satisfied.

Then the trajectory of the process may be brought to the terminal set M at a time $T=T^*_{\omega(\cdot)}(z)$ such that

$$\int_0^T\omega(T,\tau)d\tau=1,\quad T<+\infty.$$

The proof relies on that of Theorem 1.

7. MODEL EXAMPLE

Consider the conflict-controlled process

$$\begin{aligned}&\ddot{x}_i+4b^2x_i=u_i,\quad x_i,y\in R^n,\quad \|u_i\|\leq 2\sigma,\quad \|\upsilon\|\leq\sigma\\ &\ddot{y}+b^2y=\upsilon\end{aligned}\tag{7.1}$$

Changing variables in this second-order system by $z_1^i=y-x_i$, $z_2^i=x_i$, $z_4^i=y$, we obtain a system of type (1.1) with

$$z_i\in R^{4n},\quad z_i=(z_1^i,z_2^i,z_3^i,z_4^i),\quad \pi_i(z_1^i,z_2^i,z_3^i,z_4^i)=z_1^i$$

After some calculations, we get

$$W(t,\tau,\upsilon)=b^{-1}\sigma|\sin 2b(t-\tau)|S-b^{-1}\upsilon|\sin b(t-\tau)|+\varepsilon\omega(t,\tau)S,\ \upsilon\in\sigma S$$

$$W(t,\tau)=\{b^{-1}\sigma(|\sin 2b(t-\tau)|-|\sin b(t-\tau)|)+\varepsilon\omega(t,\tau)\}S$$

$$\xi_i(t,z_i,0)=z_1^i\cos 2bt-z_2^i(2b)^{-1}\sin 2bt+z_3^i(\cos 2bt-\cos bt)+z_4^i(b)^{-1}\sin bt$$

As the map $Q(t,\tau)$ we take

$$Q(t,\tau)=b^{-1}\sigma\{|\sin b(t-\tau)|-|\sin 2b(t-\tau)|-\varepsilon\omega(t,\tau)\}S$$

Condition 2 will hold with $Q(t,\tau)$ if

$$\int_0^t\{b^{-1}\sigma(|\sin 2b(t-\tau)|-|\sin b(t-\tau)|)+\varepsilon\omega(t,\tau)\}d\tau\geq 0\quad\text{for all}\quad t\geq 0\tag{7.2}$$

This inequality and the definition (5.1) imply certain restrictions on ε, depending on the time t. There are three possible cases

1. $t \geqslant 2\pi(3b)^{-1}$; $\varepsilon \geqslant \sigma(4b^2)^{-1}$

$$\omega(t,\tau)=\begin{cases}0, & \tau\in[0,t-2\pi(3b)^{-1}]\cup(t-\pi(2b)^{-1},t]\\ 4b\{|\sin b(t-\tau)|-|\sin 2b(t-\tau)|\}, & \tau\in[t-2\pi(3b)^{-1},t-\pi(2b)^{-1}]\end{cases}$$

2. $t\in(\pi(2b)^{-1},\ 2\pi(3b)^{-1})$; $\varepsilon \geqslant (b)^{-2}\sigma(-\cos bt-\cos^2 bt)$

$$\omega(t,\tau)=\begin{cases}0, & \tau\in(t-\pi(2b)^{-1},t]\\ b(|\sin b(t-\tau)|-|\sin 2b(t-\tau)|)-(-\cos bt-\cos^2 bt)^{-1}, & \tau\in[0,t-\pi(2b)^{-1}]\end{cases}$$

3. $t\in(0,\ \pi(2b)^{-1})$; $\varepsilon \geqslant 0$; $\omega(t,\ \tau)=(t)^{-1}$.

Theorem 3 implies that the time required to bring a trajectory of the process (1.1) to the terminal set, that is, $T=T_{\omega(\cdot)}(z)$, is finite, provided condition 7 holds for the initial states of the process and the parameters T and ε satisfy the above constraints.

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